CHAPTER 1.1

5 The Shapley Value

5.1 The Problem of the Commons and Two Examples

A commons is a technology used jointly by a given set of agents; the problem of the commons is to organize fairly and efficiently the exploitation of this technology. The microeconomic approach to distributive justice puts this problem at the top of its agenda: the Shapley value is an axiomatic solution to a simple model of the commons. Indeed, no systematic discussion of the commons problem was possible until the tools of (cooperative as well as noncooperative) game theory became available fifty years or so ago.

Joint ventures requiring coordinated action of partners with heterogeneous expertise are typical commons. The partners contribute their labor input and share the output (profit) generated by the enterprise. Examples include musical ensembles (example 2.4), law firms, and fishing or farming cooperatives.

The interesting question of distributive justice is to assess fairly the productive contributions from the various partners: recall the musical duo, example 2.4, where the two musicians are not equally famous. Similarly the fishermen may use various fishing techniques, with different impact on the future stock of fish; the partners in a law firm bring different kinds of expertise that are unequally scarce, and so on.

The sharing of joint costs falls squarely within the scope of the commons problem. In a cost-sharing model the agents demand certain services that are jointly produced by the technology (the commons), and they must share fairly the total cost of meeting these demands. A typical example (see example 5.6) is access to a network; each agent wants to be hooked to the central server but the connection cost is not uniform: some agents live near the server and need only a short cable, some agents are close to one another, another source of savings, and so on. This chapter and the next one are entirely devoted to cost-sharing problems like this one, and to the "dual" surplus-sharing problems where each agent contributes some productive input and the question is to share the resulting total output.

In terms of the general principles stated in section 2.1, the focus of this chapter is almost exclusively on the interpretation of reward: What is a fair assessment of individual responsibilities in the formation of total cost (or surplus)? Compensation is entirely absent from the discussion, and we always assume equal exogenous rights.[1] Fitness is not an issue in most of the chapter, where we assume inelastic demand of output or supply of input. That is to say, willingness to pay for the output or reservation values for providing the input play no role. The only exception is section 5.4, a prelude to the models of chapters 6 and 7. In the next chapter, by contrast, fitness is a paramount concern and the simultaneous pursuit of fitness and reward is the heart of the discussion.

1. Nevertheless, the axiomatic discussion of section 5.5 can be extended to accommodate asymmetric rights.

Example 5.1 Joint Venture: Example 2.4 Revisited The formal model is identical to that of example 2.4, namely two agents must share a given amount of some divisible commodity, and the division takes only into account two individual characteristics. The interpretation is quite different: the goal is to share the cost of providing a certain service to our two agents, and the individual parameters are the stand-alone costs, namely the cost of providing service to one agent alone.

Teresa and David share an office space and need to connect their computer to the network. Teresa needs a small capacity link for which the company charges c_1, whereas David needs a larger one that costs c_2, $c_1 < c_2$. There is a single cable outlet in the office, and in order to connect both of them, the company must install an additional outlet at cost δ. Thus the total bill to equip both Teresa and David is $c_{12} = c_1 + c_2 + \delta$. We call $c_i, i = 1, 2$, the stand-alone costs of our two agents: if David is out of the picture, no extra outlet is needed and Teresa will pay c_1.

Formally we have a distribution problem as in section 2.2 where a bad (cost c_{12}) must be shared and total burden exceeds the sum of individual liabilities. Which one of our three basic solutions—proportional, equal benefits, and uniform gains—if any, should we use?

Suppose that the company is running a promotional campaign for the small capacity link that Teresa needs, so that her stand-alone cost is $c_1 = 0$. In this configuration the proportional solution is highly unappealing because it charges the entire cost $c_2 + \delta$ to David, when surely Teresa should bear a share of the mutual externality δ.

The uniform gains solution—which should be called uniform costs in the context of our example—is even worse in that it seeks to equalize cost shares irrespective of the difference in stand-alone costs. For instance, if $c_1 = 0$ as above, the solution charges $y_1 = \delta$ to Teresa and $y_2 = c_2$ to David as long as $\delta \leq c_2$; it charges $\frac{1}{2}(\delta + c_2)$ to both whenever $\delta \geq c_2$. The former is unpalatable because David contributes nothing to the cost δ of the mutual externality. The latter is too, because Teresa becomes responsible for half of David's stand-alone cost c_2.

The equal surplus solution is the only sensible way to share costs in this context, since c_i is clearly a separable cost. It simply splits equally the nonseparable cost δ, and charges $y_1 = c_1 + (\delta/2)$, $y_2 = c_2 + (\delta/2)$.

Now we change the story to one where the cost of connecting Teresa and David is smaller than the sum of their two stand-alone costs. This is called a deficit configuration in section 2.2, and a subadditive cost function in this chapter: $c_{12} < c_1 + c_2$. In the previous story the cost function is superadditive: $c_{12} > c_1 + c_2$; see section 5.3.

The company charges a fee δ_i to set up a link, and this fee increases with the capacity of the link. Here $\delta_1 < \delta_2$ as Teresa needs less capacity than David.

In addition the consumer must pay a flat fee δ for the technician's visit: this fee is the same no matter how many links the technician sets up in his visit. By joining their orders, Teresa and David save one fixed fee. The stand-alone costs are $c_i = \delta_i + \delta$ for $i = 1, 2$, and the total cost is $c_{12} = \delta_1 + \delta_2 + \delta < c_1 + c_2$.

The uniform costs solution is as unappealing as above, for it ignores the difference between δ_1 and δ_2.[2] The proportional solution splits the cost-saving δ in proportion to the stand-alone costs $\delta_i + \delta$, which is an unpalatable compromise for exactly the same reasons as above. For instance, if δ_1 and δ are comparable, but δ_2 is much larger than both, Teresa gets essentially no rebate from her stand-alone costs.[3]

The uniform savings solution (i.e., the uniform losses solution of section 2.2) is the only sensible solution in the subadditive cost case. It splits the cost-savings δ equally between David and Teresa: $y_1 = \delta_1 + (\delta/2)$, $y_2 = \delta_2 + (\delta/2)$.

The discussion of example 5.1 suggests a general cost-sharing method, based on the computation of $n + 1$ numbers if the number of agents sharing the commons is n. Let c_i be agent i's stand-alone cost and c_N be the total cost of serving the whole population N. We compute individual cost shares by the equal surplus/uniform cost-saving methods of section 2.2. Thus, if the costs are superadditive, $c_N > \sum_j c_j$, each agent i pays her stand-alone cost c_i plus a surcharge equal to her fair share of the cost externality $c_N - \sum_j c_j$. If costs are subadditive, $c_N < \sum_j c_j$, everyone pays her stand-alone costs minus a common rebate, or pays nothing at all if this difference is negative:

$$\begin{aligned} c_N \geq \sum_j c_j &\Rightarrow y_i = c_i + \frac{1}{n}\left(c_N - \sum_j c_j\right) \qquad \text{for } i = 1, 2, \ldots, n \\ c_N \leq \sum_j c_j &\Rightarrow y_i = (c_i - \mu)_+ \text{ where } \sum_j (c_j - \mu)_+ = c_N \qquad \text{for } i = 1, \ldots, n \end{aligned} \tag{1}$$

These cost shares are simple and intuitive, and in example 5.1 they deliver the correct solution. In the case of a two-person problem, the cost shares (1) take the simple form of an equal rebate for both users, provided that we make the reasonable assumption $c_i \leq c_{12}$ for $i = 1, 2$; namely serving both agents cannot be cheaper than serving only one. In the subadditive case, this assumption implies that the common rebate μ is below c_i for $i = 1, 2$. Therefore, in both cases—superadditive and subadditive—the cost shares are simply

$$y_1 = \tfrac{1}{2}(c_{12} + c_1 - c_2), \quad y_2 = \tfrac{1}{2}(c_{12} + c_2 - c_1) \tag{2}$$

2. This is provided that $\delta \geq \delta_2 - \delta_1$. When $\delta \leq \delta_2 - \delta_1$, the solution is even worse: it charges her stand-alone cost $\delta_1 + \delta$ to Teresa, and David gets the full saving of one fix fee—he pays δ_2.

3. Splitting the cost saving δ in proportion to the capacity costs δ_i would give essentially all the rebate to Teresa under the same premises, which is an equally unjustified outcome.

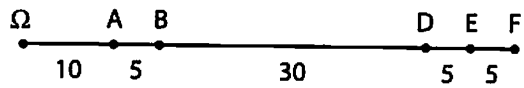

Figure 5.1
Mail distribution: Example 5.2

Our next example shows that things are not as simple when the cost must be shared among three or more users of the commons; in fact the cost shares (1) may be altogether unacceptable.

Example 5.2 Mail Distribution Five villages share the cost of a daily mail distribution. The mail is dropped daily by an outside carrier in a certain location Ω. The villagers jointly hire a local distributor who picks the mail from Ω, delivers it to the five villages, where he picks the outgoing mail, and goes back to Ω. We neglect all sorting costs (the mail is dropped at Ω in five presorted packages). The local distributor's charge is proportional to the distance he must travel daily, and the price is $1 per kilometer.

The five villages are located along the single road starting at Ω and passing successively through A, B, D, E, and F. Distances, in kilometers, are indicated on figure 5.1. Thus the daily tour from Ω to F and back costs $110. The problem is to divide it fairly among our five customers.

The stand-alone costs—costs of delivering mail to agent i only—are as follows:

$$c_A = 20, \quad c_B = 30, \quad c_D = 90, \quad c_E = 100, \quad c_F = 110$$

Therefore the formula (1) gives $\mu = 63.3$ and the following cost shares:

$$x_A = x_B = 0, \quad x_D = 26.7, \quad x_E = 36.7, \quad x_F = 46.7$$

This is obviously too soft on agents A and B who should bear a positive share of total cost!

We note that a division of the total cost 110 in proportion to the stand-alone costs above is plausible in this numerical example. However, in section 5.3 we show that the proportional solution may give unreasonable cost shares in a similar example with different distances; see example 5.4.

A simple separation argument leads to a genuine division of costs, which the Shapley value also recommends (as shown in the next section). The idea is to consider each interval such as BD separately and to split the corresponding fraction of total cost only among those agents who are responsible for it. For instance, the cost of covering the interval $[E, F]$ should be imputed to F alone. By the same token, the cost of $[D, E]$ should be split equally between E and F for this cost must be covered as soon as any of them receives mail, that of $[B, D]$ should be split three ways among D, E, and F, and so on. Hence we have the following cost shares:

$$x_A = \tfrac{1}{5}20 = 4, \quad x_B = x_A + \tfrac{1}{4}10 = 6.5, \quad x_D = x_B + \tfrac{1}{3}60 = 26.5$$

$$x_E = x_D + \tfrac{1}{2}10 = 31.5, \quad x_F = x_E + 10 = 41.5$$

The cost structure of example 5.2 appears in many contexts. The line may represent an irrigation ditch from the source O (river) to its end point E, and we must share the maintenance cost of the canal (taken to be proportional to its length) among the different farms, $A, B, \ldots$ located along the canal.

More generally, consider the cost of building the capacity of a common facility. The length of a runway increases in the size of the planes that use it, the depth of a harbor increases in the size of ships, or the cost of a network increases with the bandwidth of a link. In each case agent i requires a capacity that costs c_i to build, and the stand-alone cost of building the capacity required by the set of agents S is

$$C(S) = \max_{i \in S} c_i \tag{3}$$

For the technology (3) the separation argument of example 5.2 is easily adapted. Order the agents by increasing capacities, say $c_1 \leq c_2 \leq \cdots \leq c_n$. Note that the cost of serving S never exceeds c_{n-1} if S does not contain agent N, and always surpasses c_{n-1} by $c_n - c_{n-1}$ if S does contain this agent. Therefore assign the cost of increasing capacity from the level required by agent $n-1$ to that required by agent n, to agent n only. Split similarly the cost $c_{n-1} - c_{n-2}$ of increasing capacity from level $n-2$ to level $n-1$, equally among agents $(n-1)$ and n, and so on. The final cost shares are as follows:

$$\begin{aligned} x_1 &= \frac{1}{n}c_1, \quad x_2 = x_1 + \frac{1}{n-1}(c_2 - c_1), \quad x_3 = x_2 + \frac{1}{n-2}(c_3 - c_2) \\ x_n &= c_n - \left(\frac{1}{2}c_{n-1} + \frac{1}{6}c_{n-2} + \cdots + \frac{1}{n(n-1)}c_1\right) \end{aligned} \tag{4}$$

5.2 The Shapley Value: Definition

The basic model of the commons that is the subject of the current chapter was introduced more than fifty years ago in von Neumann and Morgenstern's *Theory of Games*, and is known in the jargon of that theory as the model of cooperative games with transferable utility. In the cost-sharing interpretation, the model specifies the set $N = \{1, 2, \ldots, n\}$ of agents who each want one unit of "service," and for each nonempty subset S of N (also called the coalition S of agents) a stand-alone cost $C(S)$ of serving the (agents in) coalition S.

For instance, in example 5.2, "service" is mail delivery and $C(S)$ is the cost of the smallest tour passing all i in S (ignoring the agents in $N\backslash S$ altogether). Thus the cost function C itself is the commons, the technology shared by all agents. The problem is to divide fairly the cost $C(N)$ of serving everyone when fairness is meant to reward the responsibility of the various agents in the total cost. Unlike formula (1), the Shapley value takes into account the stand-alone costs of all coalitions S containing more than one but fewer than n agents.

In the surplus-sharing interpretation, the number $C(S)$, often denoted $v(S)$, represents the efficient revenue (measured in money, or in some other numéraire) that the agents in S can generate by some unspecified manner of cooperation. The problem is to divide total revenue $v(N)$ by taking fairly into account the revenues $v(S)$ that various coalitions generate when standing alone. Two fundamental examples are the commons model of chapter 6, where $v(S)$ takes into account both the benefits and costs of production when the agents in S use the commons efficiently, and the exchange economy in chapter 7 where the agents are buyers or sellers and $v(S)$ is the total trading surplus of coalition S, meaning the net total benefit when the buyers and sellers in S trade optimally their own resources.

All examples in sections 5.1, 5.2, and 5.3 are cast in the cost-sharing framework. The examples in section 5.4 illustrate the (more subtle and more general) surplus-sharing model, as a prelude to its systematic application in chapters 6 and 7.

The Shapley value translates the Reward principle into an explicit division of $C(N)$ based on the $2^n - 1$ numbers $C(S)$, for all nonempty coalitions S. Formally this resembles the deficit or excess sharing problem of sections 2.2 through 2.4, where the division of t units of resources is based on the n numbers x_i (the claims, or demands). Yet the jump in mathematical complexity from $n + 1$ to $2^n - 1$ numbers is considerable, and the simple principles of proportionality, equal gains or losses cannot be generalized.

Example 5.3 Two Simple Three-Person Problems Each of three agents Ann, Bob, and Dave want a "service," and we have determined the following seven stand-alone costs:

$$C(A, B, D) = 120, \quad C(i) = 60 \qquad \text{for } i = A, B, D \tag{5}$$

$$C(AB) = C(AD) = 120, \quad C(BD) = 60 \tag{6}$$

Notice that we write $C(i)$ for the stand-alone cost of agent i, whereas the notation c_i was used in examples 5.1 and 5.2. The new notation is heavier but more transparent once all stand-alone costs play a role. The cost-saving $3C(i) - C(ABD) = 60$ should not be divided evenly because the cost of serving each of the three pairs ij reveals more externalities between Bob and Dave than between Ann and either Bob or Dave.

Imagine that service consists of a cable connection to the source O. Ann lives 60 kilometers to the west of O, while Bob and Dave live in the same location, 60 kilometers to the east of O. Thus Bob and Dave can share the same cable. If the cost of cable is $1 per kilometer, the pattern of stand-alone costs is precisely (5). The separation argument in example 5.2 makes clear that Ann should pay her full stand-alone cost, whereas Bob and Dave split the cost-saving (they each pay 30).

The point of the Shapley value is that we can deduce exactly the same cost shares from the seven numbers, (5) and (6), without invoking a specific representation of the problem, geographic or otherwise. The argument is that the *marginal cost* of serving Ann is 60, no matter who among Bob and Dave is or is not served:

$$C(A) = C(AB) - C(B) = C(AD) - C(D) = C(ABD) - C(BD) = 60$$

From these equalities the Shapley value assigns the cost share 60 to Ann. Since they play symmetric roles in (5) and (6), Bob and Dave split equally the remaining cost of 60.

Now we introduce what seems like a small modification of the stand-alone costs of a two-person coalitions (other costs being unchanged):

$$C(AB) = 120, \quad C(AD) = C(BD) = 60 \tag{7}$$

The coalitions $\{A, D\}$ and $\{B, D\}$ achieve a cost-saving of $60, whereas the coalition $\{A, B\}$ gets no saving whatsoever. Therefore Dave bears a larger share of responsibility for the overall saving $60. Should all this saving be passed to him, who would then pay nothing at all while Ann and Bob pay $60 each? That would be going too far, since Dave cannot get service for free when he stands alone. He needs Ann or Bob to bring about the saving, whence Ann and Bob must get some shares of it as well.

It is easy to represent the cost pattern (5) and (7) by a cable connection story. The three agents live in the same location, connected to the source O by a red cable and a blue cable. It costs $60 to rent either cable. Ann's machine (resp. Bob's) can only be connected via the blue cable (resp. the red cable). Dave's machine can use either cable, and two machines can use the same cable.

Yet the story of the red and blue cables does not help because Dave's responsibility in the cost of the red cable depends in some way of his cost share of the blue cable, and vice versa. There is no simple separation argument.

The Shapley value orders randomly Ann, Bob, and Dave, with equal probability on all six orderings, and assigns to an agent his *expected marginal cost*. For instance, the ordering B, A, D, yields the marginal costs

$$x_B = C(B) = 60, \quad x_A = C(AB) - C(B) = 60, \quad x_D = C(ABD) - C(AB) = 0$$

The six orderings and corresponding marginal costs are depicted in the following table:

	Marginal	Cost shares	
Ordering	Ann	Bob	Dave
A, B, D	60	60	0
A, D, B	60	60	0
D, A, B	0	60	60
D, B, A	60	0	60
B, D, A	60	60	0
B, A, D	60	60	0
Shapley value	50	50	20

where the last row of cost shares is the arithmetic average of the six rows above. Thus Dave keeps $\frac{2}{3}$ of cost savings 60, while Bob and Ann gets $\frac{1}{6}$ each.

In general, for a given ordering of N, the marginal cost of serving agent i is $x_i = C(S \cup \{i\}) - C(S)$, where S is the set of agents preceding i in this ordering. The Shapley value imputes to agent i the (arithmetic) *average of her marginal costs over all orderings of N*. This share is her *expected marginal cost* when one of the $n!$ orderings of N is chosen at random (and with uniform probability over all orderings).

To write a precise formula for the Shapley value requires some combinatorial notations. Given $N = \{1, 2, \ldots, n\}$, we write $\mathcal{A}_i$ for the set of coalitions not containing agent i, and $\mathcal{A}_i(s)$ for the subset of $\mathcal{A}_i$ containing the coalitions of size s (where s is a number between 0 and $n-1$); thus for $s = 0$, $\mathcal{A}_i$ is the empty set, and for $s = n-1$ it contains the single coalition $N \backslash \{i\}$. The Shapley value charges the following cost share to agent i:

$$x_i = \sum_{s=0}^{n-1} \sum_{S \in \mathcal{A}_i(s)} \frac{s!(n-s-1)!}{n!} \{C(S \cup \{i\}) - C(S)\} \tag{8}$$

In this summation the coefficient $s!(n-s-1)!/n!$ is the probability that the coalition S (of cardinality s) contains precisely all the agents preceding i in a random ordering of N. For instance, this probability equals $1/n$ if S is empty (the probability that agent i comes first in the ordering), equals $1/n$ if $S = N \backslash \{i\}$ (the probability that i comes last), equals $1/n(n-1)$ if $S = \{j\}$ (the probability that j comes first and i comes second), and so on.

The Shapley value formula is the single most influential contribution of the axiomatic approach to distributive justice. Its applications are diverse and numerous, as the examples in the next two sections and the discussion of chapters 6 and 7 demonstrate. Its normative justifications are very solid, as explained in section 5.5.

In a two-person problem, the Shapley value assigns the cost shares (2), as one sees at once by averaging marginal costs over the two orderings 1, 2 and 2, 1.

In a problem with three agents 1, 2, 3, formula (8) gives the following cost share for agent 1:

$$\begin{aligned} x_1 &= \tfrac{1}{3}C(1) + \tfrac{1}{6}(C(12) - C(2)) + \tfrac{1}{6}(C(13) - C(3)) + \tfrac{1}{3}(C(123) - C(23)) \\ &= \tfrac{1}{3}C(123) + \tfrac{1}{6}(C(12) + C(13) - 2C(23)) + \tfrac{1}{6}(2C(1) - C(2) - C(3)) \end{aligned} \tag{9}$$

Formula (9) also obtains by writing a table of marginal costs for the six ordering of 1, 2, 3 as we did in example 5.3, and averaging over the six rows.

We conclude this section by checking that in example 5.2, the Shapley value selects the very cost shares derived from the separation argument.

The cost function takes the form

$$C(S) = \max_{i \in S} c_i$$

and $c_A = 20, \quad c_B = 30, \quad c_D = 90, \quad c_E = 100, \quad c_F = 110$

Observe that in any ordering of $\{A, B, D, E, F\}$, the marginal cost of Ann is \$20 if she comes up first, and zero otherwise; that of Bob is decomposed in two parts: \$20 if he comes up first, plus \$10 if he is first among B, D, E, F (his marginal cost can be 30 or 10 or zero); that of Dave is \$20 if he is first in N plus \$10 if he is first among B, D, E, F, plus \$60 if he is first among D, E, F; and so on. Therefore the \$20 corresponding to the cost c_A are shared equally among all five agents; the next $\$10 = c_B - c_A$ are shared equally among B, D, E, F; the next $\$60 = c_D - c_B$ are shared equally among D, E, F, and so on, as in example 5.2. This argument generalizes to any cost function C taking the form (3), and gives the cost shares (4).

5.3 The Stand-alone Test and Stand-alone Core

A commons has *subadditive* costs if the production of the output (service to different agents) is cheaper for a group of agents than it is for each agent separately: the joint production brings positive externalities, cost savings that we must allocate among the participants. In the formal model $S \to C(S)$ introduced in the previous section, the subadditivity property says that for any two *disjoint* coalitions S, T, the stand-alone cost of $S \cup T$ is not higher than the sum of stand-alone costs of S and of T:

subadditivity: $C(S \cup T) \leq C(S) + C(T)$ when S and T are disjoint

Applying this property repeatedly yields an inequality that has already been discussed in examples 5.1 and 5.3: total cost is not larger than the sum of stand-alone costs:

$$C(N) \leq \sum_{i \in N} C(i)$$

Most examples discussed in this chapter involve subadditive costs (e.g., examples 5.2 through 5.8). However, the symmetric property of superadditive costs is also plausible. There the production of output involves negative externalities so that the stand alone cost of $S \cup T$ is greater (at least, not smaller) than the sum of the stand-alone costs of S and T:

superaddivity: $C(S \cup T) \geq C(S) + C(T)$ when S and T are disjoint

$$\Rightarrow C(N) \geq \sum_{i \in N} C(i)$$

Under superadditive costs, serving a group of agents is more expensive than serving each one separately.

Examples 5.1 offers a simple superadditive cost function. Many of the commons discussed in chapter 6 have superadditive costs. The typical example is a commons involving congestion, such as a pasture (example 6.2), a mine (example 6.6), or a queue (example 7.7). The entire discussion of chapter 6 is articulated around the two polar cases of increasing marginal costs (hence the superadditive cost function) and of decreasing marginal costs (hence the subadditive cost function): the two cases are important and interestingly different.

The stand-alone test is a simple fairness property directly inspired by the properties of sub- or superadditivity. It requires that everyone gets a share of the positive (resp. negative) externality created by a sub- (resp. super-) additive cost function.

Stand-alone Test

C subadditive $\Rightarrow x_i \leq C(i)$

C superadditive $\Rightarrow x_i \geq C(i)$

The test says that when the externality from joint production is of a constant sign, it should affect all the participating agents in the same direction.

Remarkably, the Shapley value meets the stand-alone test. To see this, recall the computation of the cost share x_i as the expected marginal cost $C(S \cup \{i\}) - C(S)$ of agent i, when S is the random set of agents preceding i in formula (8). Sub- (resp. super-) additivity of C gives

$$C(S \cup \{i\}) - C(S) \leq C(i), \qquad \text{resp.} \geq C(i)$$

hence the claim.

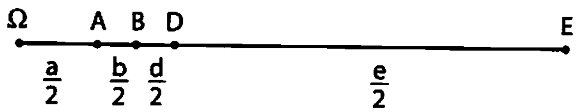

Figure 5.2
Mail distribution: Example 5.4

The stand-alone test is both compelling and easy to meet. Its generalization as the *stand-alone* core property is natural, but much more demanding.

Example 5.4 A Variant of Example 5.2 The five villages of example 5.2 are now four, and they are located along the single road starting at the source Ω and passing successively through A, B, D, and E. Distances $a/2, b/2, d/2, e/2$ correspond respectively to the intervals ΩA, AB, BD, and DE; see figure 5.2. The cost function $S \to C(S)$ is computed exactly as in example 5.2: the stand-alone cost $C(S)$ is the length of the shortest roundtrip starting at Ω and passing through all the locations in S. The Shapley division of total cost $C(N) = a + b + d + e$ is deduced from the separation argument leading to formula (4):

$$x_A = \frac{a}{4}, \quad x_B = \frac{a}{4} + \frac{b}{3}, \quad x_D = \frac{a}{4} + \frac{b}{3} + \frac{d}{2}, \quad x_E = \frac{a}{4} + \frac{b}{3} + \frac{d}{2} + e$$

Assigning cost shares in proportion to stand-alone costs is prima facie a reasonable solution:

$$x_A = \frac{a+b+d+e}{4a+3b+2d+e}a, \quad x_B = \frac{a+b+d+e}{4a+3b+2d+e}(a+b)$$

$$x_D = \frac{a+b+d+e}{4a+3b+2d+e}(a+b+d), \quad x_E = \frac{(a+b+d+e)^2}{4a+3b+2d+e}$$

The cost function is subadditive,[4] and the proportional solution obviously passes the stand-alone test because every cost share is but a fraction of one's stand-alone cost. On the other hand, some *coalition* of agents may end up paying more than stand-alone cost. For instance, if we choose $a = 10$, $b = d = 5$, $e = 50$, the proportional cost shares are computed as

$$x_A = 6.09, \quad x_B = 9.13, \quad x_D = 12.17, \quad x_E = 42.61$$

Thus $S = \{A, B, D\}$ end up paying 27.39 or 37 percent more than their stand-alone cost of 20. They are effectively subsidizing village E, which pays even less than the cost of the

4. The shortest trip stopping at every point of $S \cup T$ is shorter than any two round-trips serving all points in S and T respectively.

tour from D to E and back, for which E is solely responsible. Note that a similar argument applies to $S' = \{A, B\}$, which end up paying 15.22, or about 1.5 percent more than their stand-alone cost of 15.

The Shapley cost shares, on the other hand, never charge to a coalition S more than its stand-alone cost. This is clear from the formula above, because the agents in a coalition S pay only toward the cost of these segments that enter in the stand-alone cost of S.

The stand-alone core generalizes the stand-alone test to all coalitions of agents. Under subadditive costs, it views the stand-alone cost $C(S)$ as an upperbound on the total cost share of S; under superadditive costs, it takes this number as a lower bound on the cost imputed to S

Stand-alone Core

$$C \text{ subadditive } \Rightarrow \sum_{i \in S} x_i \leq C(S) \qquad \text{for all } S \subseteq N$$

$$C \text{ superadditive } \Rightarrow \sum_{i \in S} x_i \geq C(S) \qquad \text{for all } S \subseteq N$$

The stand-alone core property is often interpreted as a bargaining argument (private contract) when the cost is subadditive. Suppose that any coalition S can form and use freely the technology C as it pleases (in particular, agents in $N \backslash S$ cannot object to, or block in any way, S's production plan). Because the cost function C is subadditive, it is always efficient to use a single copy of the technology C to serve everyone. However, coalition S can use its stand-alone options as a disagreement outcome (as in section 3.6), rejecting accordingly any profile of cost shares (x_i) where it is charged more than $C(S)$. This argument only applies to a subadditive cost function. Even then, it must be taken with a grain of salt because the core property may prove altogether impossible to meet (see example 5.8).

In the rest of this section we show that the bite of the stand-alone core property varies wildly from one specification of the cost function to the next. In examples 5.4 and 5.5 the property cuts a large set of acceptable cost shares, among which the Shapley value is normally to be found; in another case (example 5.6) the core property cuts a very small set (even a singleton, example 5.7) of cost shares, and in this case the Shapley value is typically not in the core. Finally the stand-alone core property may be altogether too demanding, despite the sub- or superadditivity of the cost function (example 5.8).

Example 5.5 Another Mail Distribution Problem The road network depicted on figure 5.3 shows the source (post office) at Ω and the three customers Ann, Bob, and Dave. The network is more complicated than in examples 5.2 and 5.4, but the problem is the same:

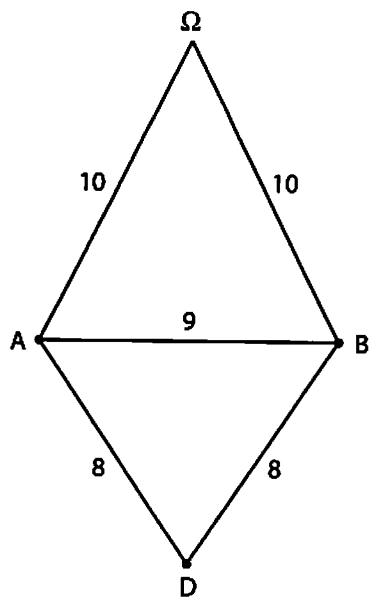

Figure 5.3
Mail distribution: Example 5.5

How should we divide the total cost of the daily tour between the three customers? The separability argument used in examples 5.2 and 5.4 does not apply here: we cannot allocate separately the cost of the various intervals of the network.

The shortest tour visiting A, B, and D goes from Ω successively to A, D, B, and back to Ω for a total cost

$$C(A, B, D) = 10 + 8 + 8 + 10 = 36$$

The stand-alone cost of A is 20 (going from Ω to A and back) as is that of B, whereas it takes a full \$36 to visit D:

$$C(A) = C(B) = 20, \quad C(D) = 36$$

Similar computations yield the stand-alone costs of two agents coalitions. For instance, the shortest tour passing through A and B costs $10 + 9 + 10 = 29$:

$$C(AB) = 29, \quad C(AD) = C(BD) = 36$$

The cost function C is subadditive as $C(ij) \leq C(i) + C(j)$ and $C(ABC) \leq C(i) + C(jk)$ for all combinations of $\{i, j, k\} = \{A, B, D\}$. The stand-alone core property places the following bounds on the cost shares x_A, x_B, x_C:

$$x_A, x_B \leq 20; \quad x_D \leq 36$$

$$x_A + x_B \leq 29; \quad x_A + x_D, x_B + x_D \leq 36$$

In view of the equality $x_A + x_B + x_D = 36$, these six inequalities can be written in a more compact way:

$$0 \leq x_A \leq 20, \quad 0 \leq x_B \leq 20, \quad 7 \leq x_D \leq 30$$

This system cuts a large subset of acceptable cost shares x, and falls very short of recommending a precise compromise between the share imputed to D, on one hand, and A, B, on the other hand.

If we respect the symmetry between A and B, we may at one extreme favor A, B against D as far as $x_A = x_B = 0$, $x_D = 36$; at the other extreme, $x_A = x_B = 14.5, x_D = 7$ is still in the stand-alone core. And we may also break the symmetry between A and B, for instance, $x_A = 0, x_B = 20, x_D = 16$.

Given the loose constraints imposed by the stand-alone core property, if is not surprising that the Shapley value passes this test. This solution is computed with the help of (9) as

$$x_A^* = x_B^* = 8.17, \quad x_D^* = 19.67$$

The next example starts from the same set of users on the same road network as example 5.5 but modifies the technology for providing service to these users. Instead of running a daily tour of all users, it is now necessary to run along the existing roads the shortest cable that will connect them to the source.

For the road network of example 5.2, depicted on figure 5.1, the length of the shortest cable joining the source Ω to the five locations A, B, D, E, F is exactly half that of a tour visiting them all; the same applies to any subset of locations. Thus the cost-sharing games "mail distribution" and "access to a network" are isomorphic (up to a factor of 2) for this road network.[5] By contrast in the case of the network of figure 5.3, the two technologies yield very different patterns of stand-alone costs; in the access problem, the stand-alone core shrinks to a small set.

Example 5.6 Access to a Network The three customers Ann, Bob, and Dave need to be connected to a network with its source at Ω. In order to connect Ω to A, B, D—or to any subset of these three—the shortest feasible cable will be used along the links of the network of figure 5.3. The cost of a connection is the total length of the cable.

5. Exercise 5.7 generalizes this observation to all "tree" networks, meaning all networks without cycles.

Thus, in order to connect Ann alone, we need a cable between Ω and A for a stand-alone cost $C(A) = 10$. In order to connect Ann and Bob, we run a cable from Ω to A and from A to B (or from Ω to B and B to A) for a total cost $C(AB) = 19$. The shortest cable connecting A, B, and D uses three links ΩA, AD, and DB (or ΩB, BD, and DA), and so on. Hence we have the cost function

$$C(A) = C(B) = 10, \quad C(D) = 18$$

$$C(AB) = 19, \quad C(AD) = C(BD) = 18, \quad C(ABD) = 26$$

As in the previous example this cost function is subadditive. But unlike in that example, the stand-alone core property cuts a very small subset of cost shares:

$$x_A \leq 10 \quad \text{and} \quad \{x_B + x_D \leq 18 \Rightarrow x_A \geq 8\}$$

where the implication follows from the budget balance condition $x_A + x_B + x_D = 26$. Thus x_A and, by symmetry, x_B are between 8 and 10. Similarly

$$x_A + x_B \leq 19 \Rightarrow x_D \geq 7, \quad x_A, x_B \geq 8 \Rightarrow x_D \leq 10$$

To sum up, a triple of cost shares x_A, x_B, x_D is in the stand-alone core if (and only if) it meets the following system:

$$8 \leq x_A, x_B \leq 10, \quad 7 \leq x_D \leq 10, \quad x_A + x_B + x_D = 26$$

The cost-sharing most advantageous to Ann and Bob (and treating them equally) is $x_A = x_B = 8, x_D = 10$; the least advantageous is $x_A = x_B = 9.5, x_D = 7$. A good compromise is $x_A = x_B = 9, x_D = 8$.

The logic of the Shapley value solution is very different, and indeed the cost shares it recommends do not meet the stand-alone core property:

$$x_A^* = x_B^* = 7.5, \quad x_D^* = 11$$

In our next example the stand-alone core contains a single set of cost shares, yet this unique allocation is not convincing.

Example 5.7 Example 5.3 Continued Consider the three-person cost-sharing example given by (5) and (7), which we repeat for convenience:

$$C(ABD) = 120, \quad C(i) = 60 \qquad \text{for } i = A, B, D$$

$$C(AB) = 120, \quad C(AD) = C(BD) = 60$$

These costs are subadditive. The Shapley value was computed as $x_A = x_B = 50, x_D = 20$. It fails the stand-alone core property by virtue of an objection of $S = \{A, D\}$ (or $\{B, D\}$): they can ignore Bob and split their own stand-alone cost of 60 as $x'_A = 45$, $x'_D = 15$, from which they both benefit. Now Bob can offer an even better deal to Dave, say $x''_B = 50$, $x''_D = 10$: this deal is feasible to Bob and Dave standing alone, and better for Bob than being left in the cold.

The bidding for Dave's cooperation does not stop here: Ann can offer an even better deal to Dave (e.g., $x'''_A = 55$, $x'''_D = 5$), and so on. The only resting point of the bidding war is when Dave has extracted the entire surplus: indeed, the unique vector of cost shares in the stand-alone core is $x_A = x_B = 60$, $x_D = 0$. This allocation may be plausible if Dave plays Ann against Bob, who never think of colluding against Dave. But it is not a plausibly fair division of the cooperative surplus, of which all the credit cannot go to Dave.

The next example is a subadditive cost function where the stand-alone core property is logically impossible.

Example 5.8 Buying a Software Ann, Bob, and Dave want to purchase software to meet certain word-processing needs. There is no shortage of software on the market, but not all are compatible with either of their computers, nor do they fill all their needs.

After carefully studying the market, our partners have located four software products:

Software product	Satisfactory for	Cost
X	Ann, Dave	\$800
Y	Bob, Dave	\$900
Z	Ann, Bob	\$1,000
E	Ann, Bob, Dave	\$1,700

Every other software product is dominated by one of these four products. Software S is dominated by S' if S' is not more expensive than S, if it satisfies at least the same needs, and if at least one of these two comparisons is strict. The cheapest software meeting Ann's needs costs \$800, which is Ann's stand-alone cost. Similarly $C(B) = 900$ and $C(D) = 800$.

Computing the Shapley value with the help of (9), we find that

$$x^*_A = 550, \quad x^*_B = 650, \quad x^*_D = 500$$

On the other hand, the stand-alone core property yields a logically impossible system of three inequalities and one equality:

$$x_A + x_B \leq 1{,}000, \quad x_A + x_D \leq 800, \quad x_B + x_D \leq 900$$

and

$$x_A + x_B + x_D = 1{,}700$$

Adding the three inequalities yields $x_A + x_B + x_D \leq 1{,}350$, a contradiction.

A natural solution in the spirit of these (out of reach) inequalities is this profile of cost shares where each inequality is violated by the same amount, namely

$$x_A + x_B - 1{,}000 = x_A + x_D - 800 = x_B + x_D - 900$$

This system—together with the constraint that total cost is \$1,700—yields the cost shares:

$$x_A = 567, \quad x_B = 667, \quad x_D = 467$$

which are not too different from the Shapley value, although the spread $x_B - x_D$ has increased to \$200.

In cooperative game theory the two ideas of the Shapley value and the stand-alone core have been studied for general cost or surplus functions $S \to C(S)$ (cooperative games with transferable utility). The feasibility of the stand-alone core property and the relation between the Shapley value and the stand-alone core have been investigated with full mathematical generality.

Finally, a couple of solutions selecting, for any cost function, a central point within the stand-alone core (or in the spirit of the stand-alone inequalities if the core is empty) have been constructed: example 5.8 provides an illustration.

Our last example is meant to remind us of one great advantage of the Shapley value, namely that it applies equally well to a cost function that is neither sub- nor superadditive. For such a cost function, even the stand-alone test ceases to make sense.

Example 5.9 Location of a Post Office We modify example 5.2 by allowing the five agents to choose the location of the post office anywhere on the road network (as in examples 3.4 and 3.8) and the cost to be shared is that of the daily delivery tour starting from the post office and passing through all relevant customers.

Thus in the network of example 5.2 (figure 5.1) any location between A and F is efficient: the corresponding tour costs $C(ABDEF) = 90$. Similarly $C(ABDE) = 80$, $C(BE) = 70$, and so on. This cost function is neither sub- nor superadditive because

$$C(ABDE) = 80 > 10 + 10 = C(AB) + C(DE)$$

$$C(ABDE) = 80 < 70 + 70 = C(AD) + C(BE)$$

Thus the logic of the stand-alone core does not apply.

The Shapley value, on the other hand, suggests a judicious way to cut through the thorny pattern of externalities. Direct computation of this solution as the expected marginal cost is tedious—there are 120 orderings of five agents—but an argument based on the additivity property of the value (section 5.5) delivers the answer almost at once.

Consider the cost 60 of the interval BD (each interval is traveled on twice). In a random ordering of the five agents if will be imputed to one of A, B if (and only if) one of D, E, F is drawn first, and to one of D, E, F if (and only if) one of A, B is drawn first. As the agents A, B are equal in this subproblem (i.e., the costsharing of the interval BD), they receive an equal expected share, and D, E, F are treated equally as well. Thus the cost of interval BD is divided as

A, B each pay $\frac{1}{2} \cdot \frac{3}{5} \cdot 60 = \$18$

D, E, F each pay $\frac{1}{3} \cdot \frac{2}{5} \cdot 60 = \$8$

Similar computations for each of the four intervals give

Intervals	A	B	D	E	F	Total cost
AB	8	0.5	0.5	0.5	0.5	10
BD	18	18	8	8	8	60
DE	1.33	1.33	1.33	3	3	10
EF	0.5	0.5	0.5	0.5	8	10
Shapley value	27.83	20.33	10.33	12	19.5	90

Exercise 5.7 generalizes this decomposition argument.

5.4 Stand-alone Surplus

We illustrate the versatility of the surplus-sharing model, defined by a pair (N, v) where the function v associates to every coalition S in N a "surplus" $v(S)$. To interpret $v(S)$, we go through the Gedank experiment where the agents in S cooperate, and use efficiently the resources they control. This results in a net benefit $v(S)$, the stand-alone surplus of coalition S that can be distributed as easily as money among the members of S. An important assumption is that individual utilities are measured in a common numéraire (e.g., cash) that is freely transferable across agents, and moreover the marginal utility of the numéraire is constant (utility is linear in money).

The key to the construction above is to define what resources the agents in S control when they stand alone. Depending on the context this control is derived from "real" property rights or from "virtual" ones. The exchange of private goods under private ownership (discussed in sections 7.1 and 7.2) is a case where the property rights are real: agents in S are free to trade among themselves; the corresponding stand-alone core property is thus interpreted as a positive statement about the stability of private contracts. On the other hand, in most instances of the commons problems (chapter 6), the stand-alone surplus represents a virtual appropriation of the technology by a certain coalition: this surplus is nevertheless relevant to the normative discussion, and the application of a general solution like the Shapley value is vindicated. In the "mail distribution" stories (examples 5.2, 5.5, and 5.9) a coalition of agents is not legally able to dismiss the agents outside the coalition, but doing so as a thought experiment is a good way to untangle the web of mutual externalities. The public contract concerns all agents in N without dropping anyone, but it is fair by reference to what would happen if some agents were dropped. This interpretation pervades this chapter and the next one.

Example 5.10 Example 5.5 Revisited As in example 5.5 the problem is to share the cost of mail delivery on the road network of figure 5.3. The difference is that we now take into account how much each agent is willing to pay for to receive mail everyday. Specifically we assume that

$$u_A = \$18, \quad u_B = \$11, \quad u_D = \$16$$

We compute the surplus function $S \to v(S)$ for each one of the seven coalitions in $\{A, B, D\}$. A single agent is not willing to pay for his own stand-alone cost ($u_A = 18 < 20 = C(A)$, etc.); therefore $v(i) = 0$ for $i = A, B, D$. Similarly any two agents' coalition is unable to achieve a positive surplus

$$u_A + u_B = 29 \leq 29 = C(AB)$$

$$u_A + u_D = 34 < 36 = C(AD)$$

$$u_B + u_D = 27 < 36 = C(BD)$$

therefore $v(ij) = 0$ for all two-person coalitions. Now efficiency commands to serve all three agents as $C(ABD) = 36 < 45 = u_A + u_B + u_D$; therefore $v(ABD) = 9$.

The surplus function is thus very simple: all three agents are equal hence the Shapley value (or any solution treating equals equally; see section 5.5) declares that each one should receive \$3 of surplus, which amounts to the following cost shares:

$$x_A = 15, \quad x_B = 8, \quad x_D = 13$$

Compare these with the cost shares in example 5.5: now Ann is paying the biggest share, whereas Bob gets a rebate. Consideration of the net benefits turns the analysis on its head.

Notice that for some other choices of the willingness to pay, the surplus-sharing approach leads to virtually the same recommendation as in example 5.5: exercises 5.2 gives an example.

Our next example illustrates an important features of the stand-alone surplus computations: a coalition S standing alone may maximize its surplus by serving only a subset of S.

Example 5.11 Example 5.6 Revisited The problem is to share the cost of a cable connecting A, B, and D to the source and following the links of the network of figure 5.3. We now assume the following willingness to pay for connection to the network:

$$u_A = \$12, \quad u_B = \$8, \quad u_D = \$12$$

Ann would pay for a connection if she was standing alone, and her net surplus would be $v(A) = 2$. Neither Bob nor Dave would buy a connection on their own: $v(B) = v(D) = 0$. Efficiency allows to connect only Ann and Dave, for a net surplus $12 + 12 - 18 = 6$ or all three agents for the same net surplus. Therefore

$$v(AD) = v(ABD) = 6$$

The coalition AB standing alone would not include Bob:

$$u_A + u_B - C(AB) = 1 < 2 = u_A - C(A)$$

hence

$$v(A) = v(AB) = 2$$

On the other hand, the coalition BD would gladly pay to connect both agents, for a surplus $v(BD) = 8 + 12 - 18 = 2$.

The surplus function v just computed is superadditive, as the reader can easily verify. The stand-alone property requires to deny any positive share of surplus to Bob:

$$\{y_A + y_D \geq 6 = y_A + y_B + y_D \quad \text{and} \quad y_B \geq 0\} \quad \text{imply} \quad y_B = 0$$

Ann and Dave share the surplus along the following guidelines:

$$2 \leq y_A \leq 4, \quad 2 \leq y_D \leq 4, \quad y_A + y_D = 6$$

The Shapley value takes a sharply different view point to distribute the six units of surplus. Bob is entitled to a positive share of surplus because he contributes a positive amount while working with Dave: $v(BD) > v(D)$. Therefore his marginal contribution is 2 whenever the ordering drawn is D, B, A. Compute the Shapley surplus shares with the help of formula (9), where cost is replaced by surplus

$$y_A = 3.33, \quad y_B = 0.33, \quad y_D = 2.33$$

If all three agents are connected, the cost shares are

$x_A = 8.67, \quad x_B = 7.67, \quad x_D = 9.67$

If only Ann and Dave are connected, Bob deserves a small cash compensation of 33cts for stepping aside and the other two agents pay:

$x'_A = 8.67, \quad x'_D = 9.67$

which covers the cost of connecting them, plus 33cts for Bob.

*5.5 Axiomatizations of the Shapley Value

The Shapley value has been axiomatically characterized in a number of ways, of which four are presented below.

A cost (or surplus) sharing problem is a pair (N, C) where N is a finite set of agents and C associates to each nonempty coalition S a real number $C(S)$. A solution associates to any such problem (N, C) a profile $x = \gamma(N, C)$ such that

$$x = (x_i)_{i \in N} \quad \text{and} \quad \sum_{i \in N} x_i = C(N)$$

The original characterization (due to Shapley) uses three axioms: *equal treatment of equals, dummy,* and *additivity.*

Equal treatment is the translation of equal exogenous rights (section 2.1) in the cost-sharing problem. We say that agents i and j are *equal* with respect to (N, C) if $C(S \cup \{i\}) = C(S \cup \{j\})$ for any set S in N containing neither i nor j (including the empty set).

Equal Treatment of Equals If i, j are equal w.r.t. (N, C), then $\gamma_i(N, C) = \gamma_j(N, C)$.

The dummy axiom is normatively the most important of the three because no other axiom conveys the reward principle. Dummy does so in a fairly convincing way, by considering an agent for which the marginal cost of joining any coalition S is zero. Say that agent i is a *dummy* in problem (N, C) if we have

$$\partial_i C(S) = C(S \cup \{i\}) - C(S) = 0 \qquad \text{for all } S \subseteq N$$

Note that for a coalition S already containing agent i, the marginal cost $\partial_i C(S)$ is zero by definition; therefore the property above has bite only for the coalitions S in $\mathcal{A}_i$ (i.e., not containing i).

The **dummy** axiom requires that a dummy agent pays nothing:

$$\{\partial_i C(S) = 0 \text{ for all } S\} \Rightarrow \gamma_i(N, C) = 0$$

The third axiom, additivity, is the most mathematically demanding, and is motivated as a decentralization property. Consider a cost function C made up of two independent costs $C^i, i = 1, 2 : C(S) = C^1(S) + C^2(S)$ for all S. For instance, if the service provided to the agent is cable TV, C^1 may represent the (one-time) cost of installing the cable connection and C^2 the variable costs of the cable company (e.g., maintenance cost of the line). The **additivity** axiom requires the cost shares to depend additively on the cost function:

$$\gamma(N, C^1 + C^2) = \gamma(N, C^1) + \gamma(N, C^2)$$

Shapley's original characterization result says that the Shapley value is the only solution meeting the three axioms equal treatment of equals, dummy, and additivity. We provide the main idea of the proof by looking, once again, at example 5.2.

We define five subproblems, the sum of which is the initial cost-sharing problem:

$C^A(S) = 20$ for all $S \neq \emptyset$

$C^B(S) = 10$ for all S s.t. $S \cap \{B, D, E, F\} \neq \emptyset$; zero otherwise

$C^D(S) = 60$ for all S s.t. $S \cap \{D, E, F\} \neq \emptyset$; zero otherwise

$C^E(S) = 10$ for all S s.t. $S \cap \{E, F\} \neq \emptyset$; zero otherwise

$C^F(S) = 10$ for all S containing F; zero otherwise

Check first that the cost function C given by (3) is precisely $C = C^A + C^B + C^D + C^E + C^F$. Next consider one of the subproblems, say C^D. Here agents A and B are dummies, and moreover D, E, and F are equal with respect to C^D. Therefore equal treatment and dummy imply that A and B pay nothing and D, E, F share equally the cost $C^D(N) = 60$. Repeating this argument, we find that the cost of C^A is shared equally among all agents, that of C^B among B, D, E, F, and so on. In turn the additivity property yields the cost shares computed in examples 5.2.

The next characterization of the Shapley value replaces the dummy and additivity axiom by a single property called

Marginalism For any two games (N, C^1), (N, C^2) and any agent i,

$$\{\partial_i C^1(S) = \partial_i C^2(S) \text{ for all } S\} \Rightarrow \{\gamma_i(N, C^1) = \gamma_i(N, C^2)\}$$

This says that agent i's cost share $\gamma_i(N, C)$ depends only on the list $\partial_i C(S)$ of his marginal contributions to all coalitions S.

It is easy to check that the only marginalist and symmetric solution for two-person problems is the Shapley value (8). Indeed, such a solution takes the form

$$y_1 = f(C(1), C(12) - C(2)), \quad y_2 = f(C(2), C(12) - C(1))$$

for some function f. The budget balance gives the following equation, upon using the letter variables x, y, z for $C(1), C(2), C(12)$:

$$f(x, z-y) + f(y, z-x) = z \text{ for all } x, y, z$$

It is a simple mathematical exercise to deduce that $f(x, x') = \frac{1}{2}(x+x')$ and the announced result for two-person problems.

To sum up, the Shapley value is the only solution for cooperative games satisfying {dummy, additivity, and equal treatment} or {marginalism and equal treatment}. All of the results discussed so far involve a fixed set N of agents, also called a *fixed population*. By contrast, the next two characterizations rely on *variable population* axioms. Given a game (N, C) we denote by $(N\backslash i, C^{-i})$ the restriction of this game to the subset $N\backslash i$, namely $C^{-i}(S) = C(S)$ for all S contained in $N\backslash i$.

Equal Impact The impact of removing agent j on agent i's share is the same as that of removing agent i on agent j's share:

$$\gamma_i(N, C) - \gamma_i(N\backslash j, C^{-j}) = \gamma_j(N, C) - \gamma_j(N\backslash i, C^{-i})$$

Equal impact, unlike additivity, is a fairness statement. Additivity is a structural invariance property. Marginalism is somewhere in between.

Related to Equal Impact, we have the following axiom:

Potential There exists a real-valued function $P(N, C)$, defined for all cooperative games (N, C), such that

$$\gamma_i(N, C) = P(N, C) - P(N\backslash i, C^{-i}) \qquad \text{for all } N, i, C$$

The Shapley value is the only solution satisfying potential; it is the only solution satisfying equal impact. Both results follow an easy induction argument on the size n of N: the statements are obvious for $n = 2$ once we note that $\gamma_i(\{i\}, C) = C(i)$ and posit $P(\emptyset, C) = 0$.

Thus the latter two results are closer to providing a constructive algorithm for deriving the Shapley value than a genuine axiomatization from first principles, like the two characterizations described earlier.

We note finally that the cost-sharing methodology leading to the Shapley value can take into account unequal exogeneous rights.

If we remove the equal treatment requirement, the interesting class of *random order values* emerges. For each ordering σ of N, the σ marginal contribution solution γ^σ charges agent their marginal cost $\gamma_i^\sigma(N, C) = \partial_i C(S)$ where S is the set of agents preceding i in σ. For instance, if $\sigma = \{2, 4, 5, 1, 3\}$, we have $\gamma_4^\sigma\ (N, C) = C(42) - C(2)$; $\gamma_1^\sigma(N, C) = C(1245) - C(245)$; and so on. Each solution γ^σ meets dummy, additivity, and marginalism. The same holds true for any convex combination of these solutions provided that the coefficients of the combination are constant: these solutions are the random order values.

Each random order value computes cost shares by (1) drawing at random an ordering σ in a lottery over orderings that does not depend on the particular function C and (2) charging σ marginal contributions. The family of all the random order values is characterized by either dummy + additivity or, essentially, marginalism.

5.6 Introduction to the Literature

The normative analysis of a "value," which is a fair compromise in the kind of cost- or surplus-sharing problems discussed in this chapter, is one-half of the theory of cooperative games with transferable utility. The other half is the strategic analysis of coalition formation, and is not relevant to this book.

A number of textbook presentations of value theory are available: Owen (1982, chs. 10, 11), Moulin (1988, ch. 5), and Young (1994, ch. 5). The common theme, as in this chapter, is to contrast the additivity axiom leading to the Shapley value, with the stand-alone core requirement (interpreted as a normative principle of no subsidization). The latter leads to a value called the *nucleolus* (Schmeidler 1969), which is technically more complicated and normatively less compelling than the Shapley value; we only allude to the nucleolus, a central point in the stand-alone core, in example 5.8.

On the other hand, our choice of examples emphasizes the versatility of the cooperative game model, and in this respect it takes inspiration from an important methodological paper by Shubik (1962), and from a variety of applications to specific problems of joint costs, for instance, Thomas (1980). Example 5.2 originates in Littlechild and Owen (1973), who were the first to compute the Shapley value of the capacity cost function (3), in the problem of allocating airport landing fees. The subsequent literature on minimal cost spanning trees in networks can be viewed as a generalization of the airport game: it inspires our examples 5.5, 5.6, 5.9, and 5.11. Sharkey (1995) is an excellent survey of the relevant literature.

Many authors have contributed to the multiple axiomatic characterizations of the Shapley value reviewed in section 5.5. The seminal paper is Shapley (1953). The marginalist characterization is due to Loehman and Whinston (1974) and Young (1985); see also Chun (1989). The equal impact characterization is due to Myerson (1977), and that by the potential function to Hart and Mas-Colell (1989). The original characterization of random order values is due to Weber (1988), and the one based on marginalism to Khmelnitskaya (1999). The special relation between the Shapley value and the stand-alone core in concave cost-sharing games is due to Shapley (1971): see exercise 5.9.

Finally, a collection of essays is devoted exclusively to the Shapley value. Roth (1988) is still a useful introduction to the many applications and variants of this concept.

Exercises to Chapter 5

Exercise 5.1 Traveling Lecturer

A lecturer will visit Chicago, New York, and Washington from his home base in Boston. The cost of the round-trip of all six partial trips to a single city or a pair of cities is as follows:

Chicago	400	Chicago and New York	450
New York	300	Chicago and Washington	500
Washington	300	New York and Washington	300
Chicago, New York, and Washington	600		

a. Check that the cost function is subadditive.

b. How should the three sponsors of the trip, based in the three cities he will visit, split the total cost according to the Shapley value?

c. Show that the stand-alone core property is feasible in this example, and that the Shapley value does not meet this property.

d. We modify the cost function as follows:

Chicago and New York	400
Chicago and Washington	450
New York and Washington	300

Other stand-alone costs are unchanged. Check subadditivity, and show that now the stand-alone core is empty. Compute the Shapley value.

Exercise 5.2 Variant of Example 5.10

Assume a common willingness to pay of $18 for all three agents. Compute the stand-alone surplus function $S \to v(S)$ as in example 5.10. Check that v is superadditive and compute the division of $v(ABD)$ recommended by the Shapley value. Compare the corresponding cost shares with those found in example 5.5. Does the Shapley value surplus division meet the stand alone core property (for the superadditive function v)?

Exercise 5.3 Variant of Example 5.11

Assume a common willingness to pay \$10. Answer the same questions as in the previous exercise (where the comparison is with the cost shares found in example 5.6).

Exercise 5.4 Surplus-Sharing Variant of Example 5.8

Each agent is willing to pay \$700 for an adequate software.

Compute the superadditive surplus function $S \to v(S)$. Check, in particular, that the efficient production plan leaves Bob with no software.

Compute the Shapley value and show that it awards a cash transfer to Bob, as a compensation for stepping aside (as in example 5.11). Is the stand-alone core property (for the superadditive function v) feasible or not?

Exercise 5.5 Variant of Example 5.8

Four softwares a, b, c, d are available on the market, at a price of \$100 each. Four agents want to combine their purchase of a couple of these softwares, so as to meet their specific needs.

Software a meets the needs of Ann and Bob; software b, that of Bob and Emily; software c, that of Ann and Dave; software d, that of Ann and Emily. Thus the cheapest way to meet all individual needs is to buy b and c for \$200. The issue is to divide fairly this cost between the four agents.

a. Compute the stand-alone costs of all 14 coalitions and check the subadditivity property.

b. Compute the cost shares recommended by the Shapley value. Are they in the stand-alone core?

Exercise 5.6

Ann, Bob, and Dave share the cost of hooking up to a network. Their willingness to pay for this service is

Ann	Bob	Dave
60	50	40

The (stand-alone) costs of hooking the various subsets of agents are

$C(A) = C(B) = 50, \quad C(D) = 60$

$C(AB) = C(AD) = 70, \quad C(BD) = 60$

$C(ABD) = 100$

a. Ignoring first the willingness to pay, compute the cost shares recommended by Shapley value. Check that it is not in the stand-alone core. Show that the stand-alone core contains a unique set of cost shares and compute it.

b. From now on we take the willingness to pay into account. Compute the surplus function and show that efficiency requires serving all three agents.

c. Compute the Shapley value of the surplus function and compute the, again unique, allocation in the stand-alone core. Compare the cost shares proposed by these two solutions with the two found in question a by ignoring the willingness to pay.

Exercise 5.7 Tree Networks

A tree is a graph where all nodes are connected and there are no cycles. The agents live on certain nodes of the tree and to each edge (a link between two nodes) is attached a cost, building or maintenance cost. The tree on figure 5.4 has five agents living in different nodes and the source marked Ω. Example 5.2 is another example where the tree is a simple line.

a. Consider the mail distribution problem (as in examples 5.2 and 5.5) for the tree of figure 5.4. Check that the total cost of a tour serving all agents is twice the sum of the costs of all edges. Compute the cost shares recommended by the Shapley value, by mimicking the separability argument used in example 5.2.

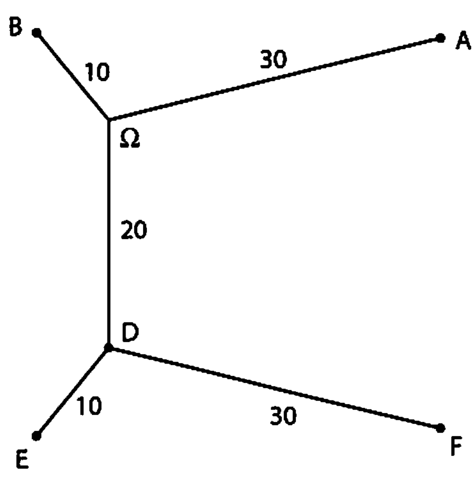

Figure 5.4
Network for exercise 5.7

b. Consider the problem "access to the network" as in example 5.6: the cost of serving a given coalition S is the total cost of the smallest subtree connecting the source to all agents in S. Check that the cost function to this problem is exactly one half of the cost function for the mail distribution problem so that the two problems are identical.

c. Suppose, as in example 5.9, that there is no assigned source and each coalition standing alone will locate the source so as to minimize the cost of a tour (or, equivalently, the cost of a subtree connecting everyone in the coalition to the source). Total cost is the same as in question a but some of the stand-alone costs are different. Compute the Shapley value, by mimicking the separability argument used in example 5.9.

***d.** Generalize the computation of the Shapley value in questions a and c to an arbitrary tree where one or several agents can live on any one of the nodes.

Exercise 5.8

Consider the network depicted in figure 5.5, showing the three agents A, B, D, the source Ω and the cost of each edge.

a. Compute the subadditive cost function of the "mail distribution" problem with source Ω as in examples 5.2 and 5.5. Compute the Shapley value profile of cost shares. Does it meet the stand-alone core property?

b. Compute the subadditive cost function of the "access to the network" problem with source Ω as in example 5.6. Show that the stand-alone core is empty.

c. Now, as in example 5.9 and question c of exercise 5.7, the agents can locate a post office anywhere on the network of figure 5.5 so as to minimize the cost of delivering mail to all

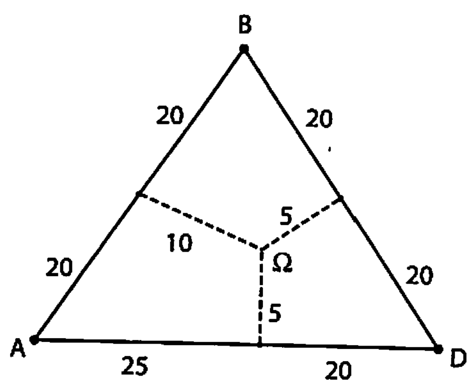

Figure 5.5
Network for exercise 5.8

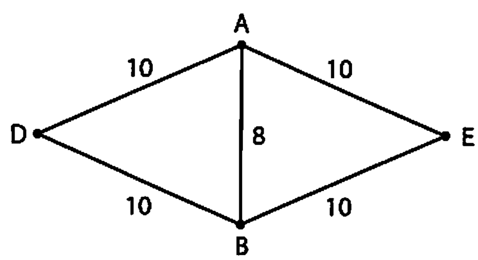

Figure 5.6
Network for exercise 5.9

of them. Note that the point Ω has no longer any special meaning, yet the inner network through Ω can be used to locate the post office. Compute the stand-alone cost function and check it is neither super- nor subadditive. Compute the Shapley value.

Exercise 5.9

Consider the four agents network depicted in figure 5.6.

a. Compute the cost function of the mail distribution problem when each coalition can choose freely the location of the source (as in example 5.9). Note that agents A, B play symmetric roles, as do D, E. Therefore one only needs to compute eight costs, corresponding to coalitions A, D, AB, DE, AD, ABD, ADE, and $ABDE$.

b. Compute the cost shares recommended by the Shapley value. (*Hint:* Use the symmetries of the problem.)

c. Consider the "access to a network" problem without a fixed source (as in example 5.9). Thus the cost of a given coalition S is that of the cheapest set of edges connecting all agents in S. Check that the cost function is neither super- nor subadditive. Compute the Shapley value.

d. Now the source Ω is fixed midway on the edge joining Ann and Bob. Compute the cost functions in the "mail distribution" (example 5.5) and "access to a network" (example 5.6) versions. Compare with the functions computed in questions a and c above. Finally compute in both cases the Shapley value.

***Exercise 5.10 Concave Cost Functions**

A cost function C is called *concave* if the marginal cost $C(S \cup i) - C(S)$ decreases as the coalition S enlarges. For all coalitions S, T,

$$S \subseteq T \Rightarrow C(S \cup i) - C(S) \geq C(T \cup i) - C(T)$$

a. For a three-person subadditive cost function, check that concavity is equivalent to three inequalities:

$$C(12) + C(23) \geq C(123) + C(2)$$

and two other inequalities by exchanging the role of the agents. Deduce that the cost function in example 5.5 is concave, but that in example 5.6 is not.

b. Show that the cost function in example 5.2 is concave. More generally, a cost function taking the form (3) is concave.

***c.** Fix an arbitrary concave cost function C and an ordering of N, say $1, 2, \ldots, n$. Show that the corresponding profile of marginal costs

$$x_1 = C(1), \quad x_2 = C(12) - C(1), \quad x_3 = C(123) - C(12), \ldots, x_n = C(N) - C(N \backslash n)$$

meets the stand-alone core property. Deduce that the Shapley value meets this property as well.

The property above explains why the stand-alone core of a concave cost function is "large." It can be shown that the stand-alone core equals the set of all convex combinations of the marginal cost vectors.